hep-th/0111161

N=1 supersymmetric sigma model with boundaries, I

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ABSTRACT

We study an $N = 1$ two-dimensional non-linear sigma model with boundaries representing, e.g., a gauge fixed open string. We describe the full set of boundary conditions compatible with $N = 1$ superconformal symmetry. The problem is analysed in two different ways: by studying requirements for invariance of the action, and by studying the conserved supercurrent. We present the target space interpretation of these results, and identify the appearance of partially integrable almost product structures.

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1 Introduction

The two-dimensional non-linear sigma model with boundaries and $N = 1$ worldsheet supersymmetry plays a prominent role in the description of open Neveu-Schwarz-Ramond (NSR) strings. Studying this model may thus help to understand those aspects of D-brane physics where the sigma model description is applicable. The model is also interesting in its own right; for instance, it is a well known fact that supersymmetric models have intriguing relations with geometry. Here we will show that the supersymmetric sigma model with boundaries naturally leads to the appearance of partially integrable almost product geometry.

The idea is to look at the non-linear sigma model with the minimal possible amount of worldsheet supersymmetry, i.e. with only one spinor parameter in the supersymmetry transformations. It turns out that even this minimal amount of supersymmetry leads to interesting restrictions on the allowed boundary conditions. These worldsheet restrictions can be reinterpreted in terms of the target space manifold, with the result that the open strings are allowed to end only on (pseudo) Riemannian submanifolds of the target space.

Special cases of the kind of conditions we present here are often adopted in the literature (e.g. [1, 2]), without derivation. The aim of this paper is to show, in a pedagogical manner, how one arrives at these conditions via systematic analysis of the action and the conserved currents. However, the conditions we find are more general than the ones usually assumed.

The paper is organised as follows. In Section 2 we consider the non-linear sigma model action defined on a (pseudo) Riemannian target space manifold. We derive the general boundary conditions that simultaneously set to zero the field variation and the supersymmetry variation of the action. We check that these conditions are compatible with the supersymmetry algebra, and then consider a special case. In Section 3 we rederive the boundary conditions using a different approach, namely requiring $N = 1$ superconformal invariance at the level of the conserved currents. In Section 4 we interpret the worldsheet conditions in terms of the target space manifold, which leads to some interesting properties of D-branes. Finally, in Section 5, we give a summary of the present work and an outline of our plans for future investigations.

2 The $\mathbf{N = 1}$ σ-model action

The first approach we take in finding the boundary conditions is to analyse the action. Field variations of the action yields boundary fields equations, which must be compatible with the vanishing of the supersymmetry variation of the action. Making a general (linear) ansatz for the relation between worldsheet fermions on the boundary, this compatibility requirement imposes restrictions on our ansatz.

2.1 Boundary equations of motion

We start from the non-linear sigma model action

$$S = \int d^2\xi d^2\theta \; D_+\Phi^\mu D_-\Phi^\nu g_{\mu\nu}(\Phi), \tag{2.1}$$

where $g_{\mu\nu}$ is a Riemannian[4] metric and we use the standard 2D superfield notation (see Appendix A). Varying S with respect to the fields Φ^μ we obtain a boundary term

$$\begin{aligned} \delta S \;\; &= \;\; i\int d\tau \Big[(\delta\psi_+^\mu\psi_+^\nu - \delta\psi_-^\mu\psi_-^\nu)g_{\mu\nu} + \\ &\quad +\delta X^\mu(ig_{\mu\nu}(\partial_{+\!\!+}X^\nu - \partial_{=}X^\nu) + (\psi_-^\nu\psi_-^\rho - \psi_+^\nu\psi_+^\rho)\Gamma_{\nu\mu\rho})\Big]_{\sigma=0,\pi} \quad , \end{aligned} \tag{2.2}$$

where $\Gamma_{\nu\mu\rho}$ is the Levi-Civita connection,

$$\Gamma_{\nu\rho\mu} = g_{\nu\sigma}\Gamma^\sigma_{\ \rho\mu} = \frac{1}{2}(g_{\nu\mu,\rho} + g_{\rho\nu,\mu} - g_{\rho\mu,\nu}). \tag{2.3}$$

We now make a general linear ansatz for the fermionic boundary conditions[5] (see [3] for a discussion of general fermionic ansätze),

$$\psi_-^\mu = \eta R^\mu_{\ \nu}(X)\,\psi_+^\nu \;\;|_{\sigma=0,\pi}, \tag{2.4}$$

where $R^\mu_{\ \nu}$ is a $(1,1)$ "tensor" which squares to one,

$$R^\mu_{\ \nu}R^\nu_{\ \rho} = \delta^\mu_{\ \rho}, \tag{2.5}$$

and we have found it convenient to introduce the parameter η which takes on the values ± 1. The property (2.5) can be justified by worldsheet parity, i.e., the theory should be invariant under interchange of ψ_+ and ψ_-. Relaxation of this property leads to the appearance of generalised torsion (i.e., the combination $B + dA$ of a B-field and gauge field); however, we focus here on a torsion-free background.[6]

We view $R^\mu_{\ \nu}$ as a formal object, for the moment avoiding to specify whether it is defined globally or only locally; such issues are discussed in Section 4. Substituting the ansatz (2.4),

[4]Hereafter, whenever we use the word "Riemannian," we mean "Riemannian or pseudo Riemannian."

[5]From now on we shall drop the notation $|_{\sigma=0,\pi}$, since all conditions in this paper are understood to hold on the boundary.

[6]The general case will be treated in a forthcoming publication [4] (see also [5].

as well as its field variation, into (2.2) yields conditions on $R^\mu_{\ \nu}$ if the variation δS is to vanish. The first such condition comes from cancelling the $\delta\psi_+^\mu$-terms and says that $R^\mu_{\ \nu}$ must preserve the metric,

$$g_{\mu\nu} = R^\sigma_{\ \mu} R^\rho_{\ \nu} g_{\sigma\rho}. \tag{2.6}$$

Note that this condition, together with (2.5), implies that $R_{\mu\nu} \equiv g_{\mu\rho} R^\rho_{\ \nu}$ is symmetric. After imposing (2.6), we are left with the δX^μ-part of (2.2),

$$\delta S = i \int d\tau \ \delta X^\mu \left[i g_{\mu\nu} (\partial_{\#} X^\nu - \partial_{=} X^\nu) + R^\nu_{\ \sigma} g_{\nu\rho} \nabla_\mu R^\rho_{\ \gamma} \psi_+^\sigma \psi_+^\gamma \right], \tag{2.7}$$

where ∇_μ is the spacetime Levi-Civita covariant derivative.

Before exploring what conditions the vanishing of (2.7) gives us, we first remark on the interpretation of $R^\mu_{\ \nu}$. The open string may either move about freely, in which case its ends obey Neumann boundary conditions; alternatively, the ends of the string may be confined to a subspace, corresponding to Dirichlet conditions. Consider a d-dimensional target space with Dp-branes, i.e., there are $d - (p+1)$ Dirichlet directions along which the field X^μ is frozen ($\partial_0 X^i = 0, i = p+1, ..., d-1$). At any given point on a Dp-brane we can choose local coordinates such that X^i are the directions normal to the brane and X^m ($m = 0, ..., p$) are coordinates on the brane. We call such a coordinate system *adapted* to the brane. In this basis the fermionic boundary conditions are [6]

$$\psi_-^m = \eta\, \psi_+^m, \qquad \psi_-^i = -\eta\, \psi_+^i,$$

whence

$$R^\mu_{\ \nu} = \begin{pmatrix} \delta^n_{\ m} & 0 \\ 0 & -\delta^i_{\ j} \end{pmatrix}. \tag{2.8}$$

(This tensor is also used in boundary state formalism, see e.g. [7, 8].) So $R^\mu_{\ \nu}$ has a clear physical interpretation: its $(+1)$ eigenvalues correspond to Neumann conditions and its (-1) eigenvalues correspond to Dirichlet conditions. Thus the general tensor $R^\mu_{\ \nu}$ represents the boundary conditions covariantly at the given point.

Given this setup of Neumann and Dirichlet directions, it is convenient to introduce the objects $P^\mu_{\ \nu}$ and $Q^\mu_{\ \nu}$ as [8]

$$P^\mu_{\ \nu} = \frac{1}{2}(\delta^\mu_{\ \nu} + R^\mu_{\ \nu}), \qquad Q^\mu_{\ \nu} = \frac{1}{2}(\delta^\mu_{\ \nu} - R^\mu_{\ \nu}). \tag{2.9}$$

Because R squares to one, $P^\mu_{\ \nu}$ and $Q^\mu_{\ \nu}$ are orthogonal projectors, defining the Neumann and Dirichlet directions, respectively, in a covariant way. They satisfy

$$P^\mu_{\ \rho} P^\rho_{\ \nu} = P^\mu_{\ \nu}, \qquad Q^\mu_{\ \rho} Q^\rho_{\ \nu} = Q^\mu_{\ \nu}, \tag{2.10}$$

and

$$Q^\mu{}_\nu P^\nu{}_\rho = P^\mu{}_\nu Q^\nu{}_\rho = 0, \qquad P^\mu_\nu + Q^\mu_\nu = \delta^\mu_\nu. \tag{2.11}$$

Continuing the analysis of the field variation (2.7), we note that since, on the boundary, the X-field is frozen along the Dirichlet directions, its corresponding variation vanishes. That is, $Q^\mu_\nu \delta X^\nu = 0$, and we may write $\delta X^\nu = P^\mu_\nu \delta X^\nu$. Thus the set of general parity invariant boundary conditions implied by the boundary equations of motion are

$$\begin{cases} \psi^\mu_- - \eta R^\mu_\nu \psi^\nu_+ = 0 \\ g_{\delta\mu} P^\mu_\nu (\partial_{\#} X^\nu - \partial_{=} X^\nu) - i P^\mu_\delta R^\nu{}_\sigma g_{\nu\rho} \nabla_\mu R^\rho{}_\gamma \psi^\sigma_+ \psi^\gamma_+ = 0 \\ Q^\mu_\nu (\partial_{=} X^\nu + \partial_{\#} X^\nu) = 0 \end{cases} \tag{2.12}$$

where R^μ_ν satisfies

$$\begin{cases} R^\mu_\nu R^\nu{}_\rho = \delta^\mu_\rho \\ g_{\mu\nu} = R^\sigma{}_\mu R^\rho{}_\nu g_{\sigma\rho} \end{cases} \tag{2.13}$$

2.2 Supersymmetry variations

In general one does not expect the boundary conditions derived as field equations to be supersymmetric, since the action is invariant under the supersymmetry algebra only up to a boundary term. To ensure worldsheet supersymmetry the boundary conditions should make the supersymmetry variation of the action vanish. The constraints on R^μ_ν implied by this condition must then be made compatible with (2.12) and (2.6).

Assuming the standard (1,1) supersymmetry variation (A.52) with supersymmetry parameter $\epsilon^+ = \eta\epsilon^-$, the variation of (2.1) yields the boundary term

$$\begin{aligned} \delta_s S &= \epsilon^- \int d\tau \Big[g_{\mu\nu}(\partial_{\#} X^\mu \psi^\nu_- - \eta \partial_{=} X^\nu \psi^\mu_+) + i g_{\mu\nu} F^\mu_{+-} (\psi^\nu_+ + \eta\psi^\nu_-) - \\ &\quad - i\psi^\mu_+ \psi^\nu_- \psi^\rho_+ \Gamma_{\mu\nu\rho} - i\eta \psi^\mu_+ \psi^\nu_- \psi^\rho_- \Gamma_{\nu\mu\rho} \Big] . \end{aligned} \tag{2.14}$$

Inserting the ansatz (2.4), $\delta_s S$ simplifies to

$$\delta_s S = \eta\epsilon^- \int d\tau \psi^\mu_+ \Big[g_{\mu\nu}(R^\nu_\sigma \partial_{\#} X^\sigma - \partial_{=} X^\nu) + 2i\eta g_{\mu\nu} P^\nu_\sigma F^\sigma_{+-} - 2i\psi^\sigma_+ \psi^\gamma_+ R^\rho{}_\sigma P^\nu{}_\mu \Gamma_{\nu\rho\gamma} \Big] . \tag{2.15}$$

We have two choices; either to stay completely off-shell, or to make use of the (algebraic) bulk field equations for the auxiliary field F.[7] If we stay off-shell and plug the conditions (2.12) into (2.15) we get a condition involving the F-field. However, this condition is not uniquely determined; for example, one can always add a term of the form $K_{\mu\nu\rho}\psi^\nu_+\psi^\rho_+$, where $K_{\mu\nu\rho}$ is symmetric either in the first two indices or in the first and the last (for instance,

[7] Note that there are no boundary contributions in deriving the F-field equations.

one could choose $K_{\mu\nu\rho} = g_{\mu\sigma}\nabla_\nu R^\sigma{}_\rho$). Therefore, staying off-shell, we do not obtain a unique form of the boundary condition for F. (Even if this F-condition were unique, we still need it to be compatible with the F-field equation on the boundary.) Hence the way to proceed is to go partly on-shell and use the F-field equation of motion,

$$g_{\mu\nu}F^\nu_{+-} + \psi^\nu_+\psi^\rho_-\Gamma_{\mu\nu\rho} = 0, \tag{2.16}$$

restricted to the boundary (i.e., with the ansatz (2.4) inserted). Substituting this in (2.15) yields the supersymmetry variation

$$\delta_s S = \eta\epsilon^- \int d\tau \psi^\mu_+ \left[g_{\mu\nu}(R^\nu{}_\sigma\partial_{+\!\!+} X^\sigma - \partial_= X^\nu)\right],$$

which, using the bosonic conditions from (2.12), reduces to

$$\delta_s S = i\eta\epsilon^- \int d\tau \psi^\mu_+\psi^\sigma_+\psi^\gamma_+ P^\delta_\mu R^\nu{}_\sigma g_{\nu\rho}\nabla_\delta R^\rho{}_\gamma. \tag{2.17}$$

$\delta_s S$ vanishes only if R^μ_ν satisfies

$$P^\delta_\mu R^\nu{}_\sigma g_{\nu\rho}\nabla_\delta R^\rho{}_\gamma + P^\delta_\sigma R^\nu{}_\gamma g_{\nu\rho}\nabla_\delta R^\rho{}_\mu + P^\delta_\gamma R^\nu{}_\mu g_{\nu\rho}\nabla_\delta R^\rho{}_\sigma = 0, \tag{2.18}$$

which, by contraction with Q, leads to

$$P^\sigma_\gamma P^\mu_\nu \nabla_{[\sigma} Q^\delta{}_{\mu]} = P^\mu_\gamma P^\nu_\sigma Q^\rho{}_{[\mu,\nu]} = 0. \tag{2.19}$$

This is the *integrability condition* for P (cf. Appendix D).

In conclusion, the boundary conditions (2.12) are consistent with worldsheet supersymmetry, in the sense discussed above, when P is integrable. The geometrical meaning of this integrability condition is discussed in Section 4.

2.3 Compatibility with the algebra

We may rephrase the question about boundary supersymmetry and study compatibility with the supersymmetry *algebra*. This means that the boundary conditions should be consistent with the supersymmetry transformation of our fermionic ansatz (2.4). Applying the transformations (A.52) to the ansatz one finds a bosonic boundary condition[8]

$$\partial_= X^\mu = R^\mu_\nu \partial_{+\!\!+} X^\nu + 2i\eta P^\mu_\nu F^\nu_{+-} - 2iR^\mu_{\nu,\sigma}P^\sigma_\rho \psi^\rho_+\psi^\nu_+. \tag{2.20}$$

[8]The set of supersymmetric boundary conditions can be written in terms of 1D superfields, see Appendix B.

Compatibility with the supersymmetry algebra is attained when the boundary field equations (2.12) satisfy (2.20). Note that (2.20) is stronger than the requirement of zero supersymmetry variation of the action; this may be seen by substituting (2.20) into (2.15) to get

$$\delta_s S = -2i\eta\epsilon^- \int d\tau\ \psi_+^\mu \psi_+^\sigma \psi_+^\gamma g_{\sigma\nu} P^\rho_{\ \mu} \nabla_\rho R^\nu_{\ \gamma}, \tag{2.21}$$

which vanishes identically because $g_{\sigma\nu} P^\rho_{\ \mu} \nabla_\rho R^\nu_{\ \gamma}$ is symmetric in σ and γ (by (2.6)).

To show that the boundary equations of motion are consistent with (2.20), we first go on-shell by using the F-field equation (2.16) restricted to the boundary, obtaining

$$\partial_= X^\mu - R^\mu_{\ \nu} \partial_+ X^\nu + 2i P^\sigma_{\ \gamma} \nabla_\sigma R^\mu_{\ \nu} \psi_+^\gamma \psi_+^\nu = 0. \tag{2.22}$$

Contraction with Q then gives the integrability condition for P, after using that $Q^\mu_{\ \nu} \partial_0 X^\nu = 0$. On the other hand, contracting (2.22) with P and using that P is integrable, we arrive at the second (bosonic) condition in (2.12). Thus we conclude that (2.12) *are the general parity invariant boundary conditions compatible with the supersymmetry algebra*, provided that R squares to one and preserves the metric, and that P is integrable.

2.4 Preservation of the metric

The properties (2.13) allow us to draw some conclusions about the form of $R^\mu_{\ \nu}$. For a generic metric $g_{\mu\nu}$ there is only one solution for R that squares to one and preserves the metric, namely $R^\mu_{\ \nu} = \delta^\mu_{\ \nu}$. Thus there can be no Dirichlet directions, i.e., this general background cannot support D-branes, or worldsheet supersymmetry is broken.

If, on the other hand, when the metric is not of a completely general form, there may be other solutions for $R^\mu_{\ \nu}$, depending on the form of $g_{\mu\nu}$. In the presence of Dp-branes, the metric must not mix Neumann and Dirichlet directions. To see why, go to adapted coordinates (X^m, X^i) at some point in the target space; then $R = \mathrm{diag}(1, -1)$ at this point, and preservation of the metric,

$$g_{\mu\nu} = R^\sigma_{\ \mu} R^\rho_{\ \nu} g_{\sigma\rho},$$

tells us that the only backgrounds that allow Dp-branes are those satisfying [1]

$$g_{in} = 0. \tag{2.23}$$

Thus we see that, at the given point, the metric must be such that the Neumann and Dirichlet directions decouple, if the strings attached to the brane are to remain supersymmetric.

There remains the important question of whether the adapted system of coordinates at a point can be extended to a neighbourhood along the Dp-brane. This is where integrability comes in; we address this issue in Section 4.

2.5 A special case

One may ask what is required for the two-fermion term in the bosonic boundary conditions in (2.12) to vanish, so that the boundary conditions take the simple form

$$\begin{cases} \psi_-^\mu = \eta R^\mu{}_\nu \psi_+^\nu \\ \partial_= X^\mu = R^\mu{}_\nu \partial_{\#} X^\nu \end{cases} \tag{2.24}$$

often assumed in the literature; see, e.g., [1]. From (2.12) it is easily seen that in addition to (2.13) one needs to impose

$$P^\rho_\mu \nabla_\rho R^\nu{}_\gamma = 0, \tag{2.25}$$

a condition that also implies P-integrability (c.f. Eqn. (2.19)).

Note that for a general metric, where we have $R^\mu{}_\nu = \delta^\mu{}_\nu$, the conditions (2.24) reduce to

$$\begin{cases} \psi_-^\mu - \eta \psi_+^\mu = 0 \\ \partial_= X^\mu - \partial_{\#} X^\mu = 0 \end{cases} \tag{2.26}$$

which corresponds to the freely moving open string, its ends satisfying Neumann conditions in all directions (corresponding to space-filling D9-branes).

3 N = 1 superconformal symmetry

The second route to finding the boundary conditions allowed by the supersymmetric sigma model involves studying the conserved currents. Here we derive these currents and the conditions they must satisfy on the boundary, showing how this yields constraints on $R^\mu{}_\nu$.

3.1 Conserved currents

We want to retain classical superconformal invariance in the presence of boundaries. To do this, the appropriate objects to study are the currents corresponding to supertranslations in $(1,1)$ superspace. These supercurrents may be derived using superspace notation (we sketch the main steps in Appendix C), and we obtain

$$T_{\#}^- = D_+ \Phi^\mu \partial_{\#} \Phi^\nu g_{\mu\nu}, \tag{3.27}$$

$$T_=^+ = D_- \Phi^\mu \partial_= \Phi^\nu g_{\mu\nu}, \tag{3.28}$$

obeying the conservation laws

$$D_+ T_=^+ = 0, \qquad D_- T_{\#}^- = 0.$$

Note that these conserved currents are defined only up to the equations of motion.

The components of the supercurrents (3.27) and (3.28) correspond to the supersymmetry current and stress tensor as follows,

$$G_+ = T^-_{+\!\!\!+}| = \psi^\mu_+ \partial_{+\!\!\!+} X^\nu g_{\mu\nu}, \tag{3.29}$$

$$G_- = T^+_{=}| = \psi^\mu_- \partial_{=} X^\nu g_{\mu\nu}, \tag{3.30}$$

$$T_{++} = -iD_+ T^-_{+\!\!\!+}| = \partial_{+\!\!\!+} X^\mu \partial_{+\!\!\!+} X^\nu g_{\mu\nu} + i\psi^\mu_+ \nabla_+ \psi^\nu_+ g_{\mu\nu}, \tag{3.31}$$

$$T_{--} = -iD_- T^+_{=}| = \partial_{=} X^\mu \partial_{=} X^\nu g_{\mu\nu} + i\psi^\mu_- \nabla_- \psi^\nu_- g_{\mu\nu}. \tag{3.32}$$

The covariant derivative acting on the worldsheet fermions is given by

$$\nabla_\pm \psi^\nu_+ = \partial_{\substack{+\!\!\!+ \\ =}} \psi^\nu_+ + \Gamma^\nu{}_{\rho\sigma} \partial_{\substack{+\!\!\!+ \\ =}} X^\rho \psi^\sigma_+, \qquad \nabla_\pm \psi^\nu_- = \partial_{\substack{+\!\!\!+ \\ =}} \psi^\nu_- + \Gamma^\nu{}_{\rho\sigma} \partial_{\substack{+\!\!\!+ \\ =}} X^\rho \psi^\sigma_-, \tag{3.33}$$

where $\Gamma^\nu{}_{\rho\sigma}$ is the Levi-Civita symbol. Moreover, the conservation laws in components acquire the following form,

$$\begin{aligned} \partial_{+\!\!\!+} G_- &= 0 \qquad\qquad & \partial_{+\!\!\!+} T_{--} &= 0 \\ \partial_{=} G_+ &= 0 \qquad\qquad & \partial_{=} T_{++} &= 0 \end{aligned}$$

To ensure the superconformal symmetry on the boundary we need to impose boundary conditions on the currents (3.29)–(3.32). To see what these conditions look like, consider the conserved charge,

$$0 = \partial_0 Q = \int d\sigma \ \ \partial_0 J^0, \tag{3.34}$$

where the current J is any of the currents G and T, and J^0 is the τ-component of J. Then current conservation, $\partial_\alpha J^\alpha = 0$, implies that

$$0 = -\int d\sigma \ \ \partial_1 J^1,$$

resulting in the boundary condition

$$J^+ - J^- = 0.$$

Applying this to our currents G and T, we arrive at the boundary conditions

$$G_+ - \eta G_- = 0, \qquad\qquad T_{++} - T_{--} = 0. \tag{3.35}$$

At the classical level, these conditions are just saying that the left-moving super-Virasoro algebra coincides with the right-moving one. We emphasize that classically the conditions (3.35) can make sense only on-shell. This means that we may (and should) make use of the field equations in our analysis of the current constraints.

3.2 Boundary conditions

To examine how the current conditions affect the choice of $R^\mu_{\ \nu}$ in the fermionic boundary conditions, we begin by rewriting the stress tensor in a suitable form. The problem is that the stress tensor has both normal and tangential fermionic derivatives. This is remedied by defining $2\nabla_0 \equiv \nabla_+ + \nabla_-$ and using the fermionic equations of motion,

$$g_{\mu\nu}(\psi_+^\mu \nabla_- \psi_+^\nu - \psi_-^\mu \nabla_+ \psi_-^\nu) = 0,$$

to rewrite $T_{++} - T_{--}$ as follows,

$$\begin{aligned} T_{++} - T_{--} &= \partial_{+\!\!+} X^\mu \partial_{+\!\!+} X^\nu g_{\mu\nu} - \partial_{=} X^\mu \partial_{=} X^\nu g_{\mu\nu} + \\ &\quad + i(\psi_+^\mu - \eta\psi_-^\mu)\nabla_0(\psi_+^\nu + \eta\psi_-^\nu) g_{\mu\nu} + \\ &\quad + i(\psi_+^\mu + \eta\psi_-^\mu)\nabla_0(\psi_+^\nu - \eta\psi_-^\nu) g_{\mu\nu}. \end{aligned} \tag{3.36}$$

The general form of the boundary conditions which satisfy (3.35) are found by again making the ansatz (2.4)

$$\psi_-^\mu = \eta R^\mu_{\ \nu} \psi_+^\nu, \qquad R^\mu_{\ \nu} R^\nu_{\ \sigma} = \delta^\mu_{\ \sigma}, \tag{3.37}$$

recalling that the last property is needed if we want worldsheet parity as a symmetry of the boundary conditions. Using (3.36) and (3.37), the conditions (3.35) may be rewritten as follows,

$$\left\{ \begin{array}{l} G_+ - \eta G_- = \psi_+^\sigma (g_{\sigma\nu}\partial_{+\!\!+} X^\nu - R^\mu_{\ \sigma} \partial_= X^\nu g_{\mu\nu}) = 0 \\ T_{++} - T_{--} = 2\partial_0 X^\rho \left(g_{\rho\nu}(\partial_{+\!\!+} X^\nu - \partial_= X^\nu) - i P^\delta_{\ \rho} R^\mu_{\ \sigma} g_{\mu\nu} \nabla_\delta R^\nu_{\ \gamma} \psi_+^\sigma \psi_+^\gamma \right) = 0 \end{array} \right. \tag{3.38}$$

We now insert $\delta^\mu_{\ \nu} = P^\mu_{\ \nu} + Q^\mu_{\ \nu}$ between $\partial_0 X^\rho$ and the parenthesis in the T-condition (assuming that there are Dirichlet directions, i.e., that there is a non-vanishing Q such that $Q^\mu_{\ \nu}\partial_0 X^\nu = 0$), and use the result in the G-condition. We thus end up with two conditions,

$$g_{\mu\nu} = R^\sigma_{\ \mu} R^\rho_{\ \nu} g_{\sigma\rho}, \qquad P^\sigma_{\ \gamma} P^\mu_{\ \nu} \nabla_{[\sigma} Q^\delta_{\ \mu]} = 0, \tag{3.39}$$

i.e., R preserves the metric and P is integrable, just as we saw in Section 2.

A remark may be in order on the fact that we obtain the individual conditions (3.39), rather than some more general condition involving all the worldsheet fields. This may be understood by recognising that, after using the fermionic boundary condition and the current conditions, we have reduced the original set of four independent fields $\left\{\psi_\pm^\mu, \partial_{\substack{+\!\!+\\=}} X^\mu\right\}$ to a set of only two independent fields, say ψ_+^μ and $\partial_{+\!\!+} X^\mu$. But since these remaining fields are truly independent, terms of different structure that appear in the current conditions must vanish separately. Thus, in the analysis of the G-condition, a term

$$\psi_+^\mu \partial_{+\!\!+} X^\nu (g_{\mu\nu} - R^\sigma_{\ \mu} R^\rho_{\ \nu} g_{\sigma\rho})$$

will appear which must vanish independently, giving the first of Eqns (3.39), and the second condition is obtained analogously.

We conclude that the complete set of boundary conditions that solve the current boundary conditions (3.35) are

$$\begin{cases} \psi_-^\mu - \eta R^\mu{}_\nu \psi_+^\nu = 0 \\ g_{\delta\mu} P^\mu_\nu (\partial_\# X^\nu - \partial_= X^\nu) - i P^\mu_\delta R^\nu{}_\sigma g_{\nu\rho} \nabla_\mu R^\rho{}_\gamma \psi_+^\sigma \psi_+^\gamma = 0 \\ P^\sigma_\gamma P^\mu_\nu \nabla_{[\sigma} Q^\delta{}_{\mu]} = 0 \\ Q^\mu_\nu (\partial_= X^\nu + \partial_\# X^\nu) = 0 \\ g_{\mu\nu} = R^\sigma{}_\mu R^\rho{}_\nu g_{\sigma\rho} \\ R^\mu{}_\nu R^\nu{}_\rho = \delta^\mu{}_\rho \end{cases} \tag{3.40}$$

These conditions are identical to the boundary conditions (2.12), (2.13) derived from the action in Section 2. In particular, it is clear that again P needs to be integrable.

As an alternative to the above procedure, one may derive the conditions (3.40) by imposing compatibility with the supersymmetry algebra on the currents in a more direct manner. That is, we can use the (on-shell) supersymmetry transformation (2.22) of the fermionic boundary conditions. The result is again that the G- and T-conditions require, in addition to the fermionic boundary condition and its supersymmetry transformation, that $R^\mu{}_\nu$ satisfy (3.39), i.e., we arrive at the boundary conditions (3.40). Note that now P-integrability may also be derived in the same way as discussed in

Section 2.3, contracting (2.22) with Q.

4 Geometric interpretation

In this section we discuss the target space interpretation of the boundary conditions we have found. All information about the boundary conditions is encoded in $R^\mu{}_\nu$, so we focus on the restrictions on $R^\mu{}_\nu$ alone. There are two ways of viewing the boundary conditions, locally and globally; either $R^\mu{}_\nu$ is defined only locally in the target space, or it is defined globally.

4.1 Locally defined conditions

We first take the local point of view, assuming that $R^\mu{}_\nu$ is defined only in some region of the target space manifold. The first boundary condition we consider is the integrability of P. Take an arbitrary point x_0^μ in the region where $R^\mu{}_\nu$ is defined. We may then write any contravariant vector dX^μ at x_0^μ as

$$dX^\mu = P^\mu_\nu dX^\nu + Q^\mu{}_\nu dX^\nu, \tag{4.41}$$

where we have used the fact that $\delta^\mu_\nu = P^\mu_\nu + Q^\mu{}_\nu$. It follows that the distribution P is defined by

$$Q^\mu{}_\nu(X)dX^\nu = 0, \tag{4.42}$$

since this leaves

$$dX^\mu = P^\mu_\nu dX^\nu. \tag{4.43}$$

The definition (4.43) leads to P-integrability, by acting on it with the exterior derivative d,

$$0 = d^2X^\mu = P^\mu_{\nu,\rho} dX^\rho \wedge dX^\nu = P^\mu_{[\nu,\rho]} P^\rho_\lambda P^\nu{}_\sigma dX^\rho \wedge dX^\nu, \tag{4.44}$$

whence

$$P^\rho_\lambda P^\nu_\sigma Q^\mu{}_{[\nu,\rho]} = 0,$$

which is the P-integrability condition (cf. Eqn (D.67)).

It turns out that P-integrability is the very condition necessary and sufficient for the differential equations (4.42) to be completely solvable in a neighbourhood. To see this, note that for $Q^\mu{}_\nu(X)dX^\nu = 0$ to be integrable, it is necessary and sufficient that in a neighbourhood,

$$0 = d(Q^\mu{}_\nu(X)dX^\nu) = \frac{1}{2} Q^\mu{}_{[\nu,\rho]} dX^\rho \wedge dX^\nu = P^\mu_\rho P^\nu{}_\sigma Q^\gamma{}_{[\mu,\nu]}. \tag{4.45}$$

If the equations (4.42) are completely integrable, then they admit rank(Q) independent solutions. Going to a coordinate basis where the equations take the form $d\tilde{X}^i = 0$, we may write the solutions as

$$\tilde{X}^i(X) = q^i, \quad i = 0, ..., \mathrm{rank}(Q) - 1, \tag{4.46}$$

where q^i are constants.

Interpreting the above result in terms of the target space manifold, we see that the coordinates (4.46) define a rank(P)-dimensional submanifold. If P has rank $p+1$, then the submanifold is $(p+1)$-dimensional, i.e., it is a Dp-brane, and $\{\tilde{X}^i\}$ is the system of coordinates adapted to the brane. Now P-integrability allows us to extend this system of coordinates to a neighbourhood (i.e., it is defined not only at one point) along the Neumann directions. Note that this result arises in a purely algebraic fashion from the requirement of minimal worldsheet supersymmetry (see Section 2).

Next we interpret the requirement that R preserve the metric,

$$g_{\mu\nu} = R^\sigma{}_\mu R^\rho{}_\nu g_{\sigma\rho}. \tag{4.47}$$

Unlike P-integrability, this property is not purely algebraic; it requires additional information, for instance conserved currents (see Section 3).

As we saw in Section 2.4, preservation of the metric implies that in coordinates adapted to the brane the metric must take a block diagonal form, $g_{\mu\nu} = \mathrm{diag}(g_{mn}, g_{ij})$, on the worldsheet boundary. Given that P is integrable, we can extend the adapted coordinate system to a neighbourhood along the brane, so that $R = \mathrm{diag}(1, -1)$ in this neighbourhood. Thus the metric is block diagonal in this neighbourhood, and the metric along the Neumann directions, g_{nm}, could in principle serve as a metric on the Dp-brane.

In conclusion, we see that requiring $N = 1$ superconformal invariance for open strings results in that they are allowed to end only on Riemannian submanifolds of the target space manifold. Of course, this implies no restrictions on the "bulk" target space; the conditions here only applies to the worldsheet boundary, telling us where the strings may end, not what the rest of the background looks like. As an aside, note that if applied to the case where both P and Q are integrable, the above discussion leads to the existence of both Dp- and D$(d-(p+1)-1)$-branes.

4.1.1 Confined geodesics

As we discussed in Section 2.5, the fermionic boundary condition and its supersymmetry transformation simplify to (2.24) when the extra requirement (2.25) is imposed. This requirement has a nice geometrical interpretation when defined in a neighbourhood of a brane. Since $\nabla_\rho R^\nu{}_\gamma = -2\nabla_\rho Q^\nu{}_\gamma$, one can rewrite (2.25) as

$$P^\rho_\mu \nabla_\rho Q^\nu{}_\gamma = 0. \tag{4.48}$$

When this condition is satisfied, P is called *parallel* [9] with respect to the Levi-Civita connection. Indeed, if P is integrable one can always construct a symmetric affine connection such that (4.48) holds.

To understand the physical meaning of (4.48), take a point x_0^μ and a vector v^μ at x_0^μ, which is contained in P (i.e., $Q^\mu_\nu v^\nu = 0$). Then the autoparallel curve with respect to the Levi-Civita connection, $v^\mu \nabla_\mu v^\nu = 0$ is uniquely determined by the initial point x_0^μ and the initial direction v^μ. The condition (4.48) ensures that the path thus determined is always contained in P. This is easily seen by inserting $v^\mu = P^\mu_\nu v^\nu$ into the curve (where now v^μ is any vector along the curve),

$$\begin{aligned} 0 = v^\mu \nabla_\mu v^\nu &= v^\mu \nabla_\mu \left(P^\nu_\sigma v^\sigma\right) \\ &= v^\mu v^\sigma \nabla_\mu P^\nu_\sigma + P^\nu_\sigma v^\mu \nabla_\mu v^\sigma \\ &= -v^\lambda v^\sigma P^\mu_\lambda \nabla_\mu Q^\nu{}_\sigma. \end{aligned}$$

Thus the autoparallel curve is compatible with the requirement that v^μ be contained in P, if (4.48) holds.

A distribution P such that autoparallel curves starting out in P always remain in P is called *geodesic*[9] [9]. Physically it means that if a geodesic starts on a Dp-brane, then it will always remain on the brane. Hence particles cannot escape from the brane.

4.2 Globally defined conditions

Turning now to global issues, we ask when it makes sense to interpret the boundary conditions (2.12), (2.13) globally and what kind of restrictions it implies for the target space. If the tensor $R^\mu_{\ \nu}$ is defined globally on the target space, and if the property $R^\mu_{\ \nu} R^\nu_{\ \rho} = \delta^\mu_{\ \rho}$ holds globally, then the target space manifold is said to admit an almost product structure $R^\mu_{\ \nu}$ (see Appendix D). If in addition $R^\mu_{\ \nu}$ preserves the metric, then the manifold is called an almost product Riemannian manifold.

Thus our boundary conditions tell us that when the target space is an almost product Riemannian manifold with the property that P is integrable, then it admits Dp-brane solutions at any point, and that these solutions respect $N = 1$ superconformal symmetry. If the target space is a locally product manifold (i.e., both P and Q are integrable), then the sigma model admits both Dp- and D$(d-(p+1))$-brane solutions at any point of the manifold, and they are all compatible with $N = 1$ superconformal invariance.

4.2.1 Warped product spacetimes

Consider again the case where P is a geodesic distribution satisfying (4.48). There is an interesting example of a geometry for which this condition is fulfilled globally, namely the warped product spacetimes [9]. Given two Riemannian manifolds $(\mathcal{M}_i, g_i)$, $i = 1, 2$, and a smooth function $f : \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}$, construct the metric $g = g_1 \oplus e^f g_2$ on $\mathcal{M}$. Then the manifold $(\mathcal{M}, g)$ is a *warped product manifold.*[10] Thus, if (4.48) is defined globally, we know that the target space is a warped product spacetime, and we also know from Section 2.5 that the only boundary conditions preserving $N = 1$ superconformal symmetry in this case are

$$\psi^\mu_- = \eta R^\mu_{\ \nu} \psi^\nu_+, \qquad \partial_= X^\mu = R^\mu_{\ \nu} \partial_{\#} X^\nu,$$

[9] This property is a weaker condition than (4.48).

[10] In the present context it is interesting that many exact solutions of Einstein equations, e.g., Schwarzschild, Robertson-Walker, Reissner-Nordström, etc., and also p-brane solutions, are examples of warped product spacetimes.

although now they need to be globally defined.

5 Discussion

In this paper we investigate what boundary conditions on a $N = 1$ sigma model are required for supersymmetry.

First, vanishing of the boundary supersymmetry variations together with invariance of the action under general variations give us a set of boundary conditions. These turn out to be compatible with the supersymmetry algebra on the boundary in that the fermionic and bosonic boundary conditions form a supersymmetry multiplet on the (auxiliary) F-field shell.

We further studied the boundary conditions required for left and right currents to agree on the boundary. The conditions derived in this way are identical to those we find using our first method.

Our boundary conditions are more general than those usually adopted in the literature and derived in a systematic manner. We believe that this make them useful.

One interesting feature of our results is the occurrence of a mixed second rank tensor $R^{\mu}{}_{\nu}$ that squares to the identity and preserves the metric. We show that the projector in the Neumann directions, formed from $R^{\mu}{}_{\nu}$, satisfies an integrability condition. This condition has the natural interpretation that it is possible to choose coordinates along a D-brane in which $R^{\mu}{}_{\nu}$ is constant and diagonal. Our boundary conditions give information about how D-branes may be embedded in spacetime and what the corresponding local geometry looks like. In particular, we show that minimal boundary supersymmetry requires open strings to end only on (pseudo) Riemannian submanifolds of the target space. This fact is usually assumed in the literature, but we have derived it in a rigorous way.

Mathematically it may also be of interest to consider the case when $R^{\mu}{}_{\nu}$ is globally defined. Then it has the geometric interpretation of an almost product structure, and we briefly discuss this.

In this paper we have only considered sigma models in a non-trivial background metric. The boundary conditions will get modified if a background antisymmetric B-field is also included [5]. We turn to this case in a future publication [4].

Other questions of interest to us are related to treating the full theory, i.e., to include the 2D supergravity fields. And, even in the gauge-fixed case, to include contributions from the ghost fields, e.g., to the currents.

Acknowledgement: We are grateful to Ingemar Bengtsson and Andrea Cappelli for discussions and comments. UL acknowledges support in part by EU contract HPNR-CT-2000-0122 and by NFR grant 650-1998368.

A (1,1) supersymmetry

Throughout the paper we use $\mu, \nu, ...$ as spacetime indices, $(\#, =)$ as worldsheet indices, and $(+, -)$ as two-dimensional spinor indices. We also use superspace conventions, where the pair of spinor coordinates of two-dimensional superspace are labelled $\theta^{\pm}$, and the covariant derivatives $D_{\pm}$ and supersymmetry generators $Q_{\pm}$ satisfy

$$\begin{aligned} D_+^2 &= i\partial_{\#}, \quad D_-^2 = i\partial_{=} \quad \{D_+, D_-\} = 0 \\ Q_{\pm} &= -D_{\pm} + 2i\theta^{\pm}\partial_{\underset{=}{\#}} \end{aligned} \tag{A.49}$$

where $\partial_{\underset{=}{\#}} = \partial_0 \pm \partial_1$. In terms of the covariant derivatives, a supersymmetry transformation of a superfield Φ is then given by

$$\begin{aligned} \delta\Phi &\equiv (\varepsilon^+ Q_+ + \varepsilon^- Q_-)\Phi \\ &= -(\varepsilon^+ D_+ + \varepsilon^- D_-)\Phi + 2i(\varepsilon^+\theta^+\partial_{\#} + \varepsilon^-\theta^-\partial_{=})\Phi. \end{aligned} \tag{A.50}$$

The components of a superfield Φ are defined via projections as follows,

$$\Phi| \equiv X, \quad D_{\pm}\Phi| \equiv \psi_{\pm}, \quad D_+D_-\Phi| \equiv F_{+-}, \tag{A.51}$$

where a vertical bar denotes "the $\theta = 0$ part of ". Thus, in components, the $(1,1)$ supersymmetry transformations are given by

$$\begin{cases} \delta X^\mu = -\epsilon^+\psi_+^\mu - \epsilon^-\psi_-^\mu \\ \delta\psi_+^\mu = -i\epsilon^+\partial_{\#}X^\mu - \epsilon^- F_{-+}^\mu \\ \delta\psi_-^\mu = -i\epsilon^-\partial_{=}X^\mu - \epsilon^+ F_{+-}^\mu \\ \delta F_{+-}^\mu = -i\epsilon^+\partial_{\#}\psi_-^\mu + i\epsilon^-\partial_-\psi_+^\mu \end{cases} \tag{A.52}$$

B 1D superfield formalism

One may view the 2D supersymmetry algebra as a combination of two 1D algebras. To see this, we rewrite the $(1,1)$ supersymmetry algebra in terms of 1D supermultiplets. Assuming that $\epsilon_+ = \eta\epsilon$ and $\epsilon_- \equiv \epsilon$, (A.52) becomes

$$\begin{cases} \delta X^\mu = -\epsilon(\eta\psi_+^\mu + \psi_-^\mu) \\ \delta(\eta\psi_+^\mu + \psi_-^\mu) = -2i\epsilon\partial_0 X^\mu \\ \delta(\psi_-^\mu - \eta\psi_+^\mu) = -2\epsilon(\eta F_{+-}^\mu - i\partial_1 X^\mu) \\ \delta(\eta F_{+-}^\mu - i\partial_1 X^\mu) = -i\epsilon\partial_\tau(\psi_-^\mu - \eta\psi_+^\mu) \end{cases} \tag{B.53}$$

Introducing a new notation for the following combinations of fields,

$$\Psi^\mu \equiv \frac{1}{\sqrt{2}}(\eta\psi_+^\mu + \psi_-^\mu), \qquad \tilde{\Psi}^\mu \equiv \frac{1}{\sqrt{2}}(\psi_-^\mu - \eta\psi_+^\mu), \qquad f^\mu \equiv \eta F_{+-}^\mu - i\partial_1 X^\mu, \tag{B.54}$$

and redefining $\epsilon = \sqrt{2}\epsilon$, the algebra (B.53) takes on the simple form

$$\begin{cases} \delta X^\mu = -\epsilon\Psi^\mu \\ \delta\Psi^\mu = -i\epsilon\partial_0 X^\mu \\ \delta\tilde{\Psi}^\mu = -\epsilon f^\mu \\ \delta f^\mu = -i\epsilon\partial_0\tilde{\Psi}^\mu \end{cases} \tag{B.55}$$

Clearly, (B.55) is a decomposition of the 2D algebra into two 1D supermultiplets. We introduce a 1D superfield notation for these multiplets,

$$K^\mu = X^\mu + \theta\Psi^\mu, \qquad S^\mu = \tilde{\Psi}^\mu + \theta f^\mu, \tag{B.56}$$

where θ is the single Grassmann coordinate of the respective 1D superspace, and the corresponding 1D superderivative is now D, satisfying $D^2 = i\partial_0$.

The fermionic boundary condition (2.4) and its supersymmetry transformation may be rewritten in terms of the 1D supermultiplets,

$$\begin{cases} \Psi^\mu = P^\mu_\nu \Psi^\nu, \\ \partial_0 X^\mu = iP^\mu_\nu \partial_0 X^\nu + \frac{1}{4}Q^\mu{}_\nu N^\nu{}_{\sigma\rho}\Psi^\sigma\Psi^\rho \\ \tilde{\Psi}^\mu = Q^\mu{}_\nu \tilde{\Psi}^\nu, \\ f^\mu = Q^\mu{}_\nu f^\nu + \frac{1}{4}Q^\mu{}_\nu N^\nu{}_{\sigma\rho}\Psi^\sigma\tilde{\Psi}^\rho \end{cases} \tag{B.57}$$

where $N^\nu{}_{\sigma\rho}$ is the Nijenhuis tensor for $R^\mu{}_\nu$, (D.68). In terms of the 1D superfields, these may be concisely written as

$$\begin{aligned} DK^\mu &= P^\mu_\nu(K) DK^\nu, \\ S^\mu &= Q^\mu{}_\nu(K) S^\nu. \end{aligned} \tag{B.58}$$

It is clear from conditions (B.58) that the multiplet (X^μ, Ψ^μ) may be thought of as living along the Neumann directions, whereas the multiplet $(\tilde{\Psi}^\mu, f^\mu)$ lives in the Dirichlet directions.

C Supercurrents

Here we briefly sketch how to derive the supercurrents that define the supersymmetry currents and stress tensor discussed in Section 3. Our derivation here includes a nonvanishing background B-field; to obtain the currents relevant to the $B = 0$ case, one just puts $B_{\mu\nu,\rho} = B_{\mu\nu} = 0$ in the result below.

We start from the superspace formulation of the non-linear sigma model, where we have promoted the worldsheet to a superspace by supplementing the ordinary worldsheet coordinates $\xi^{\pm}$ with a pair of Grassmann coordinates $\theta^{\pm}$. The model has the same form as (2.1), except now we make it locally supersymmetric by introducing the supervielbein E_M^A as well as replacing the flat superderivatives $D_{\pm}$ by covariant ones, $\nabla_{\pm}$. We have

$$S = \int d^2\xi d^2\theta \ E\nabla_+\Phi^\mu\nabla_-\Phi^\nu e_{\mu\nu}(\Phi), \tag{C.59}$$

where E is the determinant of the supervielbein. The indices run over the lightcone coordinates and their Grassmann counterparts, M and A taking values $(\#, =, +, -)$. The superfield Φ^μ is defined in Appendix A, and $e_{\mu\nu}$ is the superfield whose lowest component is the spacetime metric plus B-field, $e_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$.

Varying (C.59) with respect to the independent components of the supervielbein, we obtain an expression of the form

$$\delta S = \frac{1}{2}\int d^2\xi d^2\theta \ E\sum_A (-1)^A T_A{}^B H_B{}^A, \tag{C.60}$$

where $T_A{}^B$ are the supercurrents. We take the independent variations to be $(H_+{}^+,\ H_-{}^-,\ H_\pm{}^{\#},\ H_\pm{}^{=})$, where $H_B{}^A \equiv \delta E_A{}^M E_M^B$ [10]. Using the equations of motion,

$$\nabla_+\nabla_-\Phi^\nu g_{\mu\nu} - \frac{1}{2}\nabla_+\Phi^\rho\nabla_-\Phi^\nu\left(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} + B_{\mu\nu,\rho} + B_{\rho\mu,\nu} + B_{\nu\rho,\mu}\right) = 0,$$

as well as their $\nabla_\pm$ derivatives, all components $T_A{}^B$ vanish except $T_{\#}^-$ and $T_=^+$. To revert to the case with global supersymmetry, we reduce the covariant derivatives to flat ones again, and we get

$$T_{\#}^- = D_+\Phi^\mu\partial_{\#}\Phi^\nu g_{\mu\nu} - \frac{i}{2}D_+\Phi^\mu D_+\Phi^\nu D_+\Phi^\rho B_{\mu\nu,\rho} \tag{C.61}$$

$$T_=^+ = D_-\Phi^\mu\partial_=\Phi^\nu g_{\mu\nu} + \frac{i}{2}D_-\Phi^\mu D_-\Phi^\nu D_-\Phi^\rho B_{\mu\nu,\rho} \tag{C.62}$$

These are the supercurrents that define the supersymmetry current and stress tensor via Equations (3.29)–(3.32).

D Almost product manifolds

Here we review the relevant mathematical definitions pertaining to almost product manifolds. In our use of terminology we closely follow Yano's books [9, 11]; however, the reader should be aware that often a different terminology is used in the literature.

Let $\mathcal{M}$ be an d-dimensional manifold with a $(1,1)$ tensor $R^\mu{}_\nu$ such that, globally,

$$R^\mu{}_\nu R^\nu{}_\rho = \delta^\mu{}_\rho. \tag{D.63}$$

Then $\mathcal{M}$ is an almost product manifold with almost product structure $R^\mu{}_\nu$.

If $\mathcal{M}$ admits a Riemannian metric $g_{\mu\nu}$ such that

$$g_{\mu\nu} = R^\rho{}_\mu R^\sigma{}_\nu g_{\rho\sigma}, \tag{D.64}$$

then $\mathcal{M}$ is an almost product Riemannian manifold[11].

One may define the following two tensors on $\mathcal{M}$,

$$P^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu + R^\mu{}_\nu), \qquad Q^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu - R^\mu{}_\nu). \tag{D.65}$$

P and Q are globally defined and satisfy $R^\mu{}_\nu = P^\mu{}_\nu - Q^\mu{}_\nu$. They are orthogonal projectors, i.e.,

$$Q^\mu{}_\nu P^\nu{}_\rho = P^\mu{}_\nu Q^\nu{}_\rho = 0$$

and

$$P^\mu{}_\nu P^\nu{}_\rho = P^\mu{}_\rho, \qquad Q^\mu{}_\nu Q^\nu{}_\rho = Q^\mu{}_\rho.$$

The eigenvalues of R with respect to P and Q are $+1$ and -1, respectively. Thus P and Q define two complementary distributions such that tangent vectors v^μ on $\mathcal{M}$ with R-eigenvalue $+1$ belong to P, and vectors with R-eigenvalue -1 belong to Q,

$$P: \quad \{v^\mu : R^\mu{}_\nu v^\nu = v^\mu\}, \qquad Q: \quad \{v^\mu : R^\mu{}_\nu v^\nu = -v^\mu\}.$$

Clearly, rank(P) equals the number of $+1$ eigenvalues of R, and rank(Q) is the number of -1 eigenvalues of R. In fact, this implies that the holonomy group of an almost product Riemannian manifold is[12] $O(\mathrm{rank}(P)) \times O(\mathrm{rank}(Q))$.

At any one point x_0^μ in $\mathcal{M}$, it is always possible to find a local coordinate basis such that R, P and Q take their canonical form

$$R^\mu{}_\nu = \begin{pmatrix} \delta^m{}_n & 0 \\ 0 & -\delta^i{}_j \end{pmatrix}, \quad P^\mu{}_\nu = \begin{pmatrix} \delta^m{}_n & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^\mu{}_\nu = \begin{pmatrix} 0 & 0 \\ 0 & -\delta^i{}_j \end{pmatrix}. \tag{D.66}$$

The almost product structure R as well as the distributions P and Q may or may not be *integrable*. The integrability condition for P and Q are stated as follows.[13] P is completely

[11] This is not much of a restriction, since if the manifold allows a metric $\tilde{g}_{\mu\nu}$, then $g_{\mu\nu} = \tilde{g}_{\mu\nu} + R^\rho{}_\mu R^\sigma{}_\nu \tilde{g}_{\rho\sigma}$ will satisfy (D.64).

[12] For a *pseudo* Riemannian manifold the holonomy group is $O(1, \mathrm{rank}(P) - 1) \times O(\mathrm{rank}(Q))$.

[13] Integrability may also be expressed in terms of the Frobenius theorem.

integrable if

$$P^{\mu}_{\ \gamma}P^{\nu}_{\ \sigma}Q^{\rho}_{\ [\mu,\nu]} = \frac{1}{8}(N^{\rho}_{\ \gamma\sigma} - R^{\rho}_{\ \mu}N^{\mu}_{\ \gamma\sigma}) = \frac{1}{4}Q^{\rho}_{\ \mu}N^{\mu}_{\ \gamma\sigma} = 0 \tag{D.67}$$

where $N^{\mu}_{\ \gamma\sigma}$ is the Nijenhuis tensor for R,

$$N^{\rho}_{\ \mu\nu} = R^{\gamma}_{\ \mu}R^{\rho}_{\ [\nu,\gamma]} - R^{\gamma}_{\ \nu}R^{\rho}_{\ [\mu,\gamma]}. \tag{D.68}$$

Q is completly integrable if

$$Q^{\mu}_{\ \gamma}Q^{\nu}_{\ \sigma}P^{\rho}_{\ [\mu,\nu]} = \frac{1}{8}(N^{\rho}_{\ \gamma\sigma} + R^{\rho}_{\ \mu}N^{\mu}_{\ \gamma\sigma}) = \frac{1}{4}P^{\rho}_{\ \mu}N^{\mu}_{\ \gamma\sigma} = 0. \tag{D.69}$$

R is integrable if both Q and P are integrable, i.e., if $N^{\rho}_{\ \mu\nu} = 0$. In this case $\mathcal{M}$ is an integrable almost product manifold, also called a locally product manifold.

Integrability determines the extent to which R, P and Q may keep their canonical form in a local neighbourhood of the point x_0^{μ}. On an integrable almost product manifold, R, P and Q can always be brought to the form (D.66) in a whole neighbourhood of x_0^{μ}. However, if only one of P and Q is integrable, then the canonical form can be extended only in the corresponding directions. Thus, if P is integrable, one can adopt (D.66) along the P-directions, and similarly for Q-integrability.

In terms of transition functions on $\mathcal{M}$, R-integrability means that there is a system of coordinate neighbourhoods with coordinates X^{μ} splitting into (X^n, X^i) such that the transition functions f^{μ} are of the form $\tilde{X}^n = f^n(X^m)$ and $\tilde{X}^i = f^i(X^j)$ so that $f^n_{\ ,i} = 0$ and $f^i_{\ ,n} = 0$. This is analogous to the case of almost complex manifolds (where $R^2 = -1$); there R-integrability implies that one can find local (anti)holomorphic coordinates with (anti)holomorphic transition functions.

If, for a locally product manifold $\mathcal{M}$, $R^{\mu}_{\ \nu}$ is a covariantly constant tensor (i.e., $\nabla_{\rho}R^{\mu}_{\ \nu} = 0$ with respect to some connection), then the manifold is called a locally decomposable Riemannian manifold. The warped product manifold (see discussion at the end of Section 4) is an example of a locally product manifold which is not locally decomposable.

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